Self-organized criticality in two-variable models

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We present a cellular automaton approach involving two variables and investigate its behavior with respect to self-organized criticality (SOC). It can be seen as a generalization of the Bak-Tang-Wiesenfeld and Olami-Feder-Christensen models and exhibits SOC behavior, too. In contrast to these models it leads to a power law distribution of the cluster sizes with an exponent close to one, as it occurs in earthquakes and landsliding processes, without any tuning.

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I. INTRODUCTION

The concept of self-organized criticality (SOC) was introduced by Per Bak and his co-workers in 1987 [1–3]; it turned out to be a powerful tool for explaining the occurrence of fractal structures and $1/f$ noise in dynamic systems. A system shows SOC behavior if it tends towards a state which is stationary except for fluctuations, and where the size distribution of the related events is fractal in space and time. In analogy to critical systems in thermodynamics, this state is called ‘‘critical.’’ The fundamental difference between conventional critical systems and SOC systems is that the first ones must be tuned to become critical, while SOC systems organize themselves into the critical state. Thus, the SOC concept is suitable for explaining why fractal structures seem to be preferred in certain natural systems.

II. THE SEMINAL SOC MODELS

The basic models exhibiting SOC behavior are cellular automata on regular grids. A dynamical variable u_i is assigned to each node i of the lattice; this variable may be, e.g., a number of sand grains, a force, an energy or even a measure of fitness in a model of evolution. u_i increases through time continuously or in discrete steps. If u_i exceeds a given threshold Γ, the site i becomes unstable and relaxes. This means that u_i decreases, while a part of the loss is transferred to the neighbors of the node. If this transfer causes one of the neighbors to become unstable, it relaxes, too; this may result in avalanches of different sizes. Jensen [4] summarized these apparently fundamental criteria in the term ‘‘slowly driven, interaction dominated threshold systems.’’

The Olami-Feder-Christensen (OFC) [5] model can be seen as a prototype of such a model. Starting with some small random initial values, the continuous variables u_i grow through time at a constant rate r:

$$\partial_t u_i = r. \tag{1}$$

A site becomes unstable as soon as u_i reaches a threshold value Γ, i.e., as soon as the condition

$$u_i < \Gamma \tag{2}$$

is violated. In this case, the site i relaxes according to

$$u_i \rightarrow 0, \ \ u_j \rightarrow u_j + \alpha u_i \ \ \text{for} \ \ j \in N(i), \tag{3}$$

where α is a parameter and $N(i)$ denotes the nearest neighborhood of the site i. On a quadratic, two-dimensional lattice $N(i)$ consists of four nodes in the bulk, and two or three nodes at the boundary. After the site i has relaxed, the stability of the neighbors is checked. Those nodes which have become unstable are relaxed simultaneously according to the same rules; this procedure is repeated until all sites are stable again. Obviously, the sites affected by an avalanche form a cluster; their number is called ‘‘cluster size.’’ If the model is seen as a representation of a spatially distributed process, the cluster size measures the size of the area affected by the avalanche. In general, the cluster size is smaller than the number of relaxations taking place during an avalanche because each site may relax several times. The number of relaxation cycles needed until all nodes have become stable again can be interpreted as the duration of the avalanche; however, this duration cannot be linked with the time scale t because relaxation takes place immediately.

In the one- and two-dimensional case, this approach can be seen as a cellular automaton representation of the Burridge-Knopoff [6] model for the occurrence of earthquakes at a fault. Here, u_i corresponds to a force acting on a block in a block-spring model. This analogy restricts the parameter α according to $2d\alpha \leqslant 1$, where d is the spatial dimension. For $2d\alpha = 1$, the force is completely transferred to the neighbors in case of an instability, so that the model is conservative except for boundary effects. If $2d\alpha < 1$, a fraction $1-2d\alpha$ is lost; the model is nonconservative.

Figure 1 shows the probability density of the cluster sizes in the two-dimensional OFC model on a 64×64 grid for different values of α. The boundary sites are treated in the same way as the interior ones; the amount of u_i passing the boundary is lost. The simulations include 2×10^7 avalanches; in order to avoid artificial effects of the initial conditions only the second half of them is included in the statistics.

In the conservative case ($\alpha = \frac{1}{4}$), the probability density is a power law with an exponent of about $\frac{7}{6}$; it increases as α decreases. The empirically found Gutenberg-Richter law for

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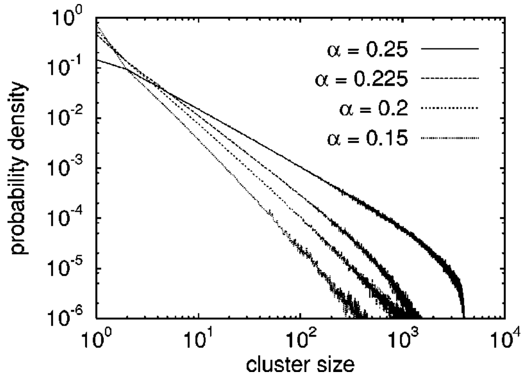

FIG. 1. Probability density of the cluster sizes in the two-dimensional OFC model on a 64×64 grid for different levels of conservation.

earthquakes [7] states exponents between 1.8 and 2.2; this behavior is achieved if α is about 0.2.

Several authors [8–11] found out that the SOC behavior of the OFC model turns into a periodic state if α decreases below a certain value; however, there is still a discussion about this critical value [4]. A second aspect that still receives attention is the sensitivity of the nonconservative OFC model against changes in the boundary conditions [4].

From a general point of view, the original Bak-Tang-Wiesenfeld (BTW) [1,2] model is quite similar to the OFC model in the conservative case. Thus, the OFC model can be seen as a generalization of the BTW model. Except for the conservation, the major difference is that the BTW model is based on a discrete variable and is driven randomly. The behavior is quite similar, although the exponents differ slightly.

III. LANDSLIDES AS A SOC PHENOMENON

In addition to earthquakes, landslides are one of the most striking phenomena exhibiting fractal magnitude statistics in earth sciences. Results from landslide mapping [12–15] show that the exponent of the probability density of landslide area is close to two and thus quite similar to that of earthquakes.

For this analogy, the idea of transferring the OFC model to landslide dynamics is tempting; but it was not successful yet. Landsliding is decisively controlled by the geometry of the land surface, especially by the slope gradient and perhaps by the second derivatives. Thus, the model variable u_i should represent these quantities; in the simplest case by a linear combination of both. This leads to a strictly conservative OFC model, as can be seen by the following argument: If $F(x_1,x_2)$ is an arbitrary function within a two dimensional domain Ω, the theorem of Gauss guarantees that the overall (integrated) value of $\partial_i F(x_1,x_2)$ only depends on the values of F at the boundaries

$$\int_\Omega \partial_i F(x_1,x_2)\,dx_1 dx_2 = \int_{\partial\Omega} F(x_1,x_2)\,n_i\,dx, \tag{4}$$

where n_i denotes the ith component of the outer normal vector. This relation holds for any reasonable discretization, so that changes in surface height at the interior nodes do not affect the sum of the u_i over the region. This implies that the loss of u_i occurring due to a landslide at any interior point is completely distributed to the neighborhood. In contrast to Eq. (3), the redistribution may be anisotropic; this results in exponents which are slightly larger than $\frac{7}{6}$, but the exponents are still far away from the observed value close to two. Thus, the BTW and OFC models are not suitable for explaining the spatial ‘‘fingerprint’’ of landslide dynamics from a quantitative point of view.

IV. TWO-VARIABLE APPROACHES

As many physical systems, landslide dynamics cannot be described by a single variable; even if the surface geometry remains constant, the risk of a landslide increases through time due to time-dependent weakening of the failure plane. This effect can be incorporated by introducing a second variable v_i and a stability criterion that depends on both u_i and v_i.

In the simplest case, both u_i and v_i grow through time uniformly and at constant rates

$$\partial_t u_i = r_u\,, \quad \partial_t v_i = r_v\,. \tag{5}$$

If we consider a slope, a uniform growth of u_i can result from slowly tilting the slope, perhaps as a consequence of tectonic forces. In the following, we restrict our considerations to regular, two-dimensional lattices.

As discussed above, the relaxation rule for u_i must be conservative. For simplicity, we only consider the isotropic case, but our computations have shown that a modest anisotropy does not affect the results. While v_i describes the state of weakening, its value should be lost completely in case of a landslide, so that the relaxation rules read

$$u_i \to 0, \quad v_i \to 0, \quad u_j \to u_j + \frac{u_i}{4} \quad \text{for} \quad j \in N(i). \tag{6}$$

There are several ways of combining the variables u_i and v_i in the stability criterion. The most basic ways are adding or multiplying both.

A. The sum approach

Let us first take a linear combination of u_i and v_i as a criterion for the stability of the node i:

$$\lambda_u u_i + \lambda_v v_i < \Gamma, \tag{7}$$

where λ_u and λ_v are positive numbers. The model can be scaled by a transformation

$$u_i := \frac{\lambda_u}{\Gamma} u_i\,, \quad v_i := \frac{\lambda_v}{\Gamma} v_i\,, \quad t := \frac{\lambda_u r_u + \lambda_v r_v}{\Gamma} t. \tag{8}$$

After this, the rule for driving the model reads

$$\partial_t u_i = \beta, \quad \partial_t v_i = 1 - \beta, \tag{9}$$

where

$$\beta = \frac{\lambda_u u_i}{\lambda_u u_i + \lambda_v v_i} \in [0,1]. \tag{10}$$

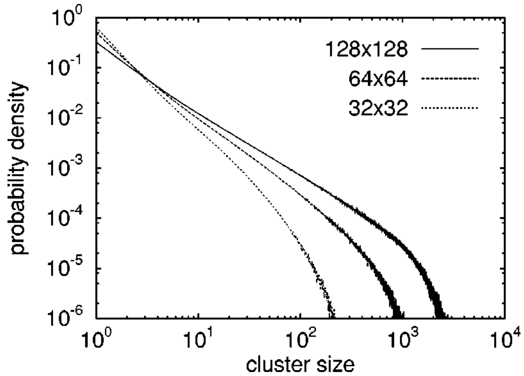

FIG. 2. Probability density of the cluster sizes in the sum approach for $\beta=0.01$ on grids of different sizes. Statistics and boundary conditions are the same as for the OFC model shown above.

The criterion for the stability of the node i now reads

$$u_i+v_i<1. \tag{11}$$

Although this approach produces power law statistics for the cluster sizes, it is not critical in the sense of SOC: Fig. 2 shows that the exponent of the power law depends on the mesh size for small values of β. In contrast, the concept of SOC only allows a cutoff at large event sizes as an effect of the finite grid size. In general, identifying such a model where the grid size affects the results strongly with a physical process is difficult.

Further simulations show that the grid size effect vanishes for larger values of β; but in this case the effect of the second variable becomes negligible and the model leads to the conservative OFC model. Thus, the sum approach is nor applicable to landslide dynamics, neither does it provide a real extension of the OFC model.

B. The product approach

Another elementary way of combining two variables to a stability criterion is

$$u_i v_i<\Gamma. \tag{12}$$

There is a fundamental difference between sum and product approach: In the sum approach, a lack of u_i can easily be compensated by an increase of v_i and vice versa. In contrast, if u_i is small, a very large value of v_i is necessary to compensate this in a product. This means that each of the variables is somehow able to inhibit instability in the product approach. Thus, this approach is reasonable for processes such as landsliding; e.g., if the surface is flat, even a very weak material remains stable.

Again, the variables and the time can be rescaled:

$$u_i := \sqrt{\frac{r_v}{\Gamma r_u}}\,u_i, \quad v_i := \sqrt{\frac{r_u}{\Gamma r_v}}\,v_i, \quad t := \sqrt{\frac{r_u r_v}{\Gamma}}\,t. \tag{13}$$

After this, the rule for driving the model reads

$$\partial_t u_i=\partial_t v_i=1, \tag{14}$$

and the criterion for the stability of the node i turns into

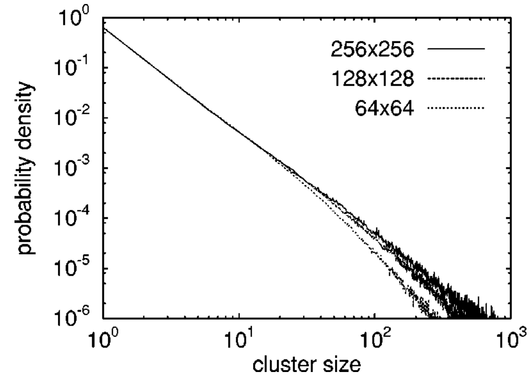

FIG. 3. Probability density of cluster sizes in the product approach on grids of different sizes.

$$u_i v_i<1. \tag{15}$$

Figures 3 and 4 show the probability densities of cluster sizes and avalanche lifetimes resulting from this approach on grids of different sizes. Statistics and boundary conditions are the same as for the OFC model shown above. Both show a power law behavior; the finite grid size only enters in form of a cutoff at large cluster sizes or lifetimes. As required for SOC, the cutoff effect is shifted towards larger events as the grid size increases.

Thus, the product approach exhibits proper SOC behavior. From a quantitative point of view, it is remarkable that the exponent of the probability density of the cluster sizes is close to two without any tuning. This leads to a fractal distribution of the cluster sizes with an exponent close to one, as it is observed in landsliding processes [12–15].

V. CONCLUSION

Motivated by the aim of relating landslide dynamics to SOC we have presented a two-variable product approach. From a basic point of view, this model is a quite straightforward extension of the fundamental BTW and OFC models which are, however, not able to reproduce the fractal size distribution of landslides quantitatively.

The model shows similar results as the BTW and OFC models, but from a quantitative point of view the spatial cluster size distributions differ: While the exponent of the BTW model is close to zero (i.e., one for the density) and the

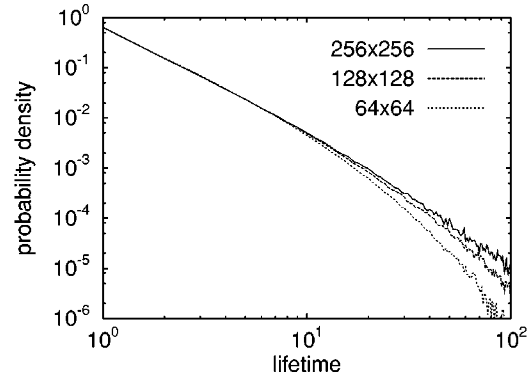

FIG. 4. Probability density of avalanche lifetimes in the product approach on grids of different sizes.

exponent of the OFC model can be tuned, the product approach leads to exponents close to 1 (i.e., two for the density) without any tuning. Within the accuracy of field measurements, this value fits well to observed landslide statistics. We suspect that some other phenomena exhibiting SOC behavior with cluster size distributions with exponents close to one might be explained by the dynamics of two variables in this sense.

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